Introduction to inverse scattering problems

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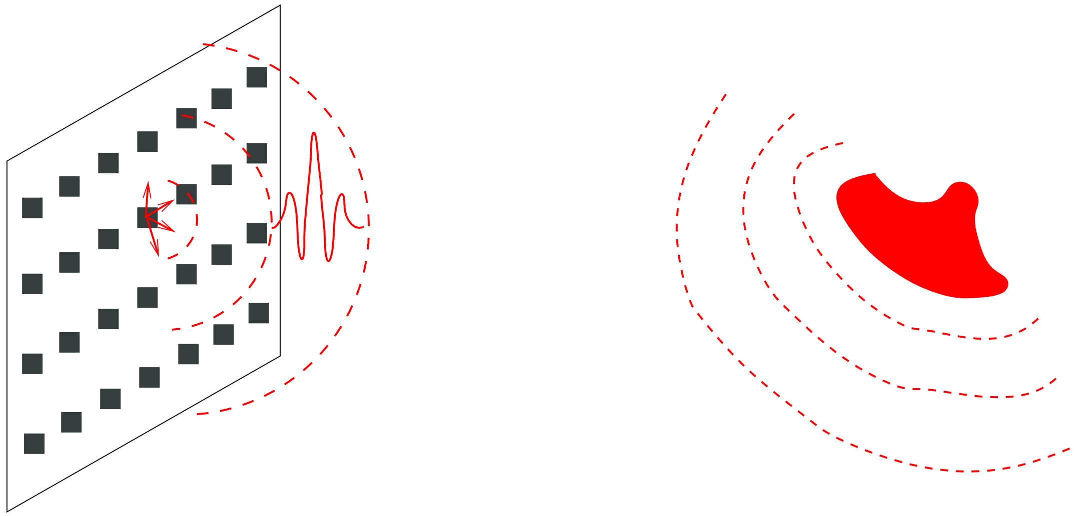


Generic setup: A collection (array) of sensors probes a medium with signals (pulses, chirps) that generate waves which are scattered by inhomogeneities. The sensors collect the scattered waves and the goal of the inversion is to estimate the medium.

Numerous applications: medical ultrasound, nondestructive evaluation of structures, radar imaging, oil exploration, etc.

- **Sound waves:** pressure $p(t,\boldsymbol{x})$ and velocity $\boldsymbol{v}(t,\boldsymbol{x})$ satisfy

$$\frac{\sigma(\boldsymbol{x})}{c(\boldsymbol{x})}\partial_t \boldsymbol{v}(t,\boldsymbol{x}) + \nabla p(t,\boldsymbol{x}) = \boldsymbol{F}(t,\boldsymbol{x})$$
$$\partial_t p(t,\boldsymbol{x}) + \sigma(\boldsymbol{x})c(\boldsymbol{x})\nabla\cdot\boldsymbol{v}(t,\boldsymbol{x}) = 0, \quad t>0,\ \boldsymbol{x}\in\mathbb{R}^3.$$

Medium modeled by acoustic impedance $\sigma(\boldsymbol{x})$ & wave speed $c(\boldsymbol{x})$.

- **Electromagnetics:** electric field $\boldsymbol{E}(t,\boldsymbol{x})$ satisfies

$$\nabla\times\nabla\times\boldsymbol{E}(t,\boldsymbol{x}) + \frac{1}{c^2(\boldsymbol{x})}\partial_t^2\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{F}(t,\boldsymbol{x}), \quad t>0,\ \boldsymbol{x}\in\mathbb{R}^3,$$

Medium with constant magnetic permeability, wave speed $c(\boldsymbol{x})$.

- $\boldsymbol{F}(t,\boldsymbol{x})$ models the excitation. Homogeneous initial conditions.

Inversion data: $p(t,\boldsymbol{x})$ or $\boldsymbol{E}(t,\boldsymbol{x})$ at the receiving sensors.

Basic (acoustic) model

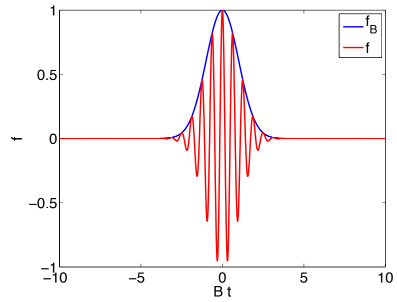
- Acoustic pressure in medium with constant density

$$\left[\frac{1}{c^2(\boldsymbol{x})}\partial_t^2 - \Delta\right] p(t,\boldsymbol{x};\boldsymbol{x}_s) = -\nabla\cdot\boldsymbol{F}(t,\boldsymbol{x}), \quad p(t,\boldsymbol{x};\boldsymbol{x}_s) \equiv 0 \text{ for } t \ll 0.$$

- Emitter is a point source: $\quad -\nabla\cdot\boldsymbol{F}(t,\boldsymbol{x}) = \delta(\boldsymbol{x}-\boldsymbol{x}_s)f(t)$

- In array imaging the signal is a pulse $\quad f(t) = e^{-i\omega_o t}B\varphi(Bt)$

display of Real$[f(t)]$



It oscillates at central frequency ω_o and is supported at $t \sim 1/B$, where $B =$ bandwidth

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t)e^{i\omega t} = \hat{\varphi}\left(\frac{\omega-\omega_o}{B}\right)$$

Why a pulse?

The length scale relations are important in inversion:

- Central wavelength $\lambda_o = \frac{2\pi c_o}{\omega_o}$.
- Distance (range) L between array and imaging scene.
- Linear size a of array aperture (may be synthetic).
- Distance c_o/B traveled by waves over pulse duration.

In radar and seismic applications: $L \gtrsim a \gg c_o/B \gg \lambda_o$

$\rightsquigarrow$ high frequency (small wavelength) regime.

As a rule, the smaller λ_o and c_o/B are, the better the imaging. The ratio a/L also plays a role.

Chirped signals and pulse compression

• Antennas have limited instantaneous power: $|f(t)|^2 \leq P_{\text{max}}$. For a signal of duration T, the emitted energy is $\leq TP_{\text{max}}$.

• The received energy is a fraction of this (partial reflection, geometrical spreading). This energy should be large to distinguish from noise $\rightsquigarrow$ Use more antennas or increase duration T.

Chirped (linear frequency modulated) signal $f(t) = e^{-i\omega_o t + i\gamma t^2}\varphi\left(\frac{t}{T}\right)$.
Assuming $\sqrt{\gamma}T \gg 1$, the Fourier transform is

$$\hat{f}(\omega) \approx \sqrt{\frac{i\pi}{\gamma}} e^{-i\frac{(\omega-\omega_o)^2}{2\gamma T}}\varphi\left(\frac{\omega_o - \omega}{2\gamma T}\right) \rightsquigarrow \quad B = \gamma T.$$

Note that $T \gg \frac{1}{\sqrt{\gamma}} = \sqrt{\frac{T}{B}} \rightsquigarrow \quad T \gg 1/B$.

Long signal is compressed to get a pulse of duration $\sim 1/B$.

- Pulse compression realized by convolving the echoes with $f(-t)$

- Why does this work?

$$
\begin{aligned}
f_c(t) &= f(t) \star f(-t) \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\hat{f}(\omega)|^2 e^{-i\omega t} \\
&\approx \frac{1}{2\gamma} \int_{-\infty}^{\infty} d\omega \left| \varphi\left(\frac{\omega_o - \omega}{2B}\right) \right|^2 e^{-i\omega t}
\end{aligned}
$$

Example: if $\varphi(s) = 1_{[-1/2,1/2]}(s)$ we get

$$
f_c(t) = Te^{-i\omega_o t}\text{sinc}(Bt).
$$

We have transformed the signal with duration $T \gg 1/B$ to a pulse oscillating at frequency ω_o and support $t \sim 1/B$.

Forward (data) model

- The acoustic pressure is a superposition of time harmonic waves

$$p(t, \boldsymbol{x}; \boldsymbol{x}_s) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \widehat{p}(\omega, \boldsymbol{x}; \boldsymbol{x}_s) e^{-i\omega t}$$

satisfying the Helmholtz equation

$$\Delta \widehat{p}(\omega, \boldsymbol{x}; \boldsymbol{x}_s) + \frac{\omega^2}{c^2(\boldsymbol{x})} \widehat{p}(\omega, \boldsymbol{x}; \boldsymbol{x}_s) = -\widehat{f}(\omega)\delta(\boldsymbol{x} - \boldsymbol{x}_s)$$

with outgoing (radiation) conditions.

- We have access to $\widehat{p}(\omega, \boldsymbol{x}_r; \boldsymbol{x}_s)$ for $|\omega - \omega_o| \leq B$, $r = 1, \ldots, N_r$ and $s = 1, \ldots, N_s$.

- Forward model relates unknown $c(\boldsymbol{x})$ to these measurements.

This is a very complicated mapping!

What can we invert for?

- **Inversion model** uses separation of scales:

$$\frac{1}{c^2(\boldsymbol{x})} = \frac{1}{c_o^2(\boldsymbol{x})}[1 + \rho(\boldsymbol{x}) + \mu(\boldsymbol{x})]$$

$c_o(\boldsymbol{x})$ = smooth, determines kinematics of waves (travel times).

$\rho(\boldsymbol{x})$ = rough part, is the reflectivity that we wish to determine.

$\mu(\boldsymbol{x})$ models small variations at small scale (clutter), that may have a cumulative scattering effect on the wave.

- **What can we estimate?**

- Smooth $c_o(\boldsymbol{x})$ (velocity analysis): Travel time tomography (many applied papers, theory of Uhlmann, Stefanov, Vasy). Differential semblance optimization (Symes). Here $c_o =$ constant.

- Reflectivity ρ (imaging problem).

- Clutter cannot be estimated $\rightsquigarrow$ random model of uncertain μ.

Born approximation: linear forward model

- For $c_o =$ constant and neglecting clutter,

$$\left(\Delta + k^2\right)\widehat{p}(\omega, \boldsymbol{x}; \boldsymbol{x}_s) = -\widehat{f}(\omega)\delta(\boldsymbol{x} - \boldsymbol{x}_s) - \frac{\omega^2}{c_o^2}\rho(\boldsymbol{x})\widehat{p}(\omega, \boldsymbol{x}; \boldsymbol{x}_s).$$

- Inverting the Helmholtz operator using Green's function

$$\widehat{G}(\omega, \boldsymbol{x}, \boldsymbol{x}_s) = \frac{e^{i\omega\tau(\boldsymbol{x},\boldsymbol{x}_s)}}{4\pi|\boldsymbol{x} - \boldsymbol{x}_s|}, \qquad \tau(\boldsymbol{x}, \boldsymbol{x}_s) = \frac{|\boldsymbol{x} - \boldsymbol{x}_s|}{c_o},$$

we get

$$\widehat{p}(\omega, \boldsymbol{x}_r; \boldsymbol{x}_s) = \widehat{f}(\omega)\widehat{G}(\omega, \boldsymbol{x}_r, \boldsymbol{x}_s) + \frac{\omega^2}{c_o^2}\int d\boldsymbol{y}\, \rho(\boldsymbol{y})\widehat{p}(\omega, \boldsymbol{y}; \boldsymbol{x}_s)\widehat{G}(\omega, \boldsymbol{x}_r, \boldsymbol{y}).$$

- Born (single scattering) approximation

$$\widehat{p}(\omega, \boldsymbol{y}; \boldsymbol{x}_s) \rightsquigarrow \widehat{f}(\omega)\widehat{G}(\omega, \boldsymbol{y}, \boldsymbol{x}_s)$$

is justified for ρ of small support, like a point reflector.

Data model

- Additive noise, single scattering model

$$D(t, \boldsymbol{x}_r, \boldsymbol{x}_s) = p(t, \boldsymbol{x}_r; \boldsymbol{x}_s) + \mathcal{N}(t, \boldsymbol{x}_r, \boldsymbol{x}_s)$$

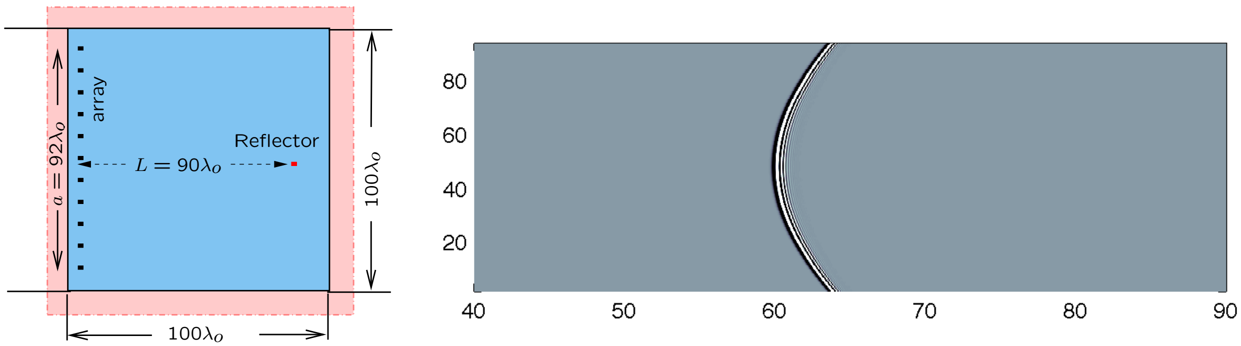
Typically $\mathcal{N}(t, \boldsymbol{x}_r, \boldsymbol{x}_s)$ is Gaussian, uncorrelated over the sensors.

- By time windowing the direct wave from $\boldsymbol{x}_s$ to $\boldsymbol{x}_r$,

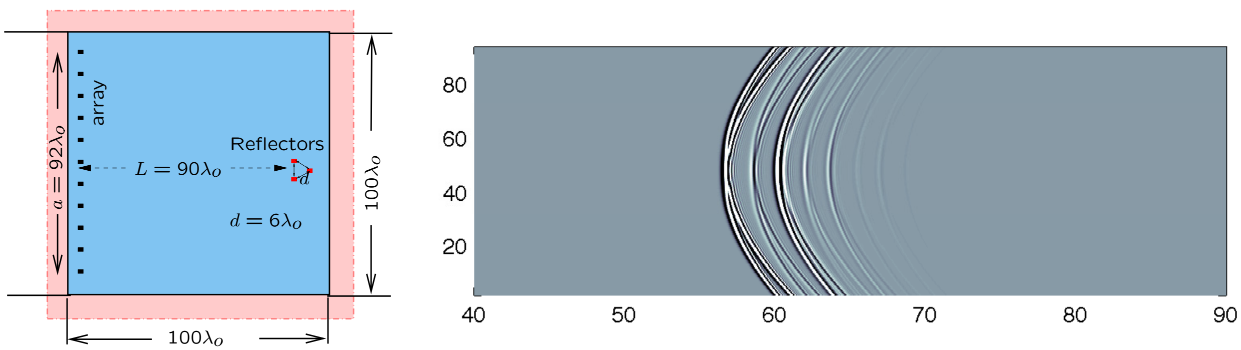
$$\begin{aligned} p(t, \boldsymbol{x}_r; \boldsymbol{x}_s) &\rightsquigarrow \int \frac{d\omega}{2\pi} \frac{\omega^2}{c_o^2} \widehat{f}(\omega) \int d\boldsymbol{y}\, \rho(\boldsymbol{y}) \frac{e^{i\omega\tau(\boldsymbol{y}, \boldsymbol{x}_s)}}{4\pi|\boldsymbol{y} - \boldsymbol{x}_s|} \frac{e^{i\omega\tau(\boldsymbol{y}, \boldsymbol{x}_r)}}{4\pi|\boldsymbol{y} - \boldsymbol{x}_r|} \\ &= -\frac{1}{(4\pi c_o)^2} \int d\boldsymbol{y}\, \rho(\boldsymbol{y}) \frac{f''\Big[t - \tau(\boldsymbol{y}, \boldsymbol{x}_s) - \tau(\boldsymbol{y}, \boldsymbol{x}_r)\Big]}{|\boldsymbol{y} - \boldsymbol{x}_s||\boldsymbol{y} - \boldsymbol{x}_r|}. \end{aligned}$$

- Depending on support of ρ and size of array, geometrical spreading factors may be approximated by a constant.

Numerically simulated data*



Scattered wave by point reflector plotted vs. time on abscissa and receiver location on ordinate. Center sensor emits pulse $f(t)$.



Multiply scattered echos among point reflectors are ignored.

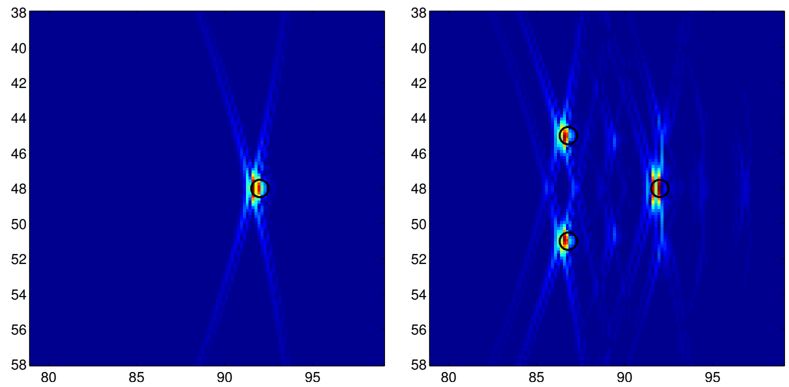
*Simulations by Chrysoula Tsogka.

Image formation - Reverse time (Kirchhoff) migration

- The imaging function

$$\mathcal{I}(\vec{\boldsymbol{y}}) = \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} D\Big(\tau(\boldsymbol{y}, \boldsymbol{x}_s) + \tau(\boldsymbol{y}, \boldsymbol{x}_r), \boldsymbol{x}_r, \boldsymbol{x}_s\Big)$$

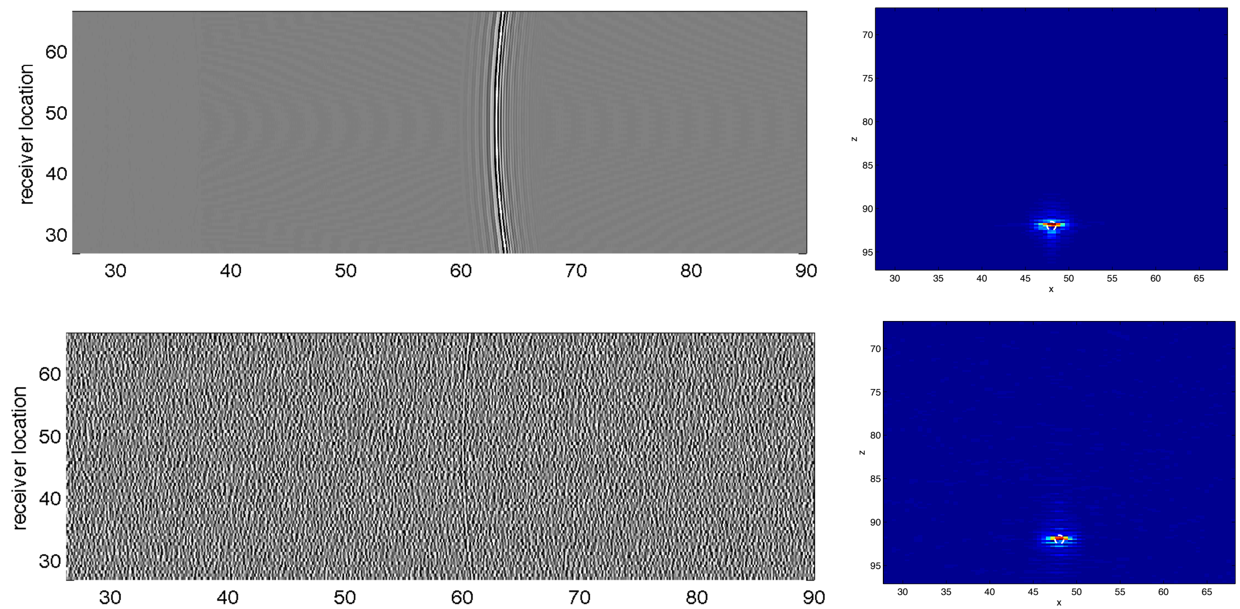
is expected to peak at points $\vec{\boldsymbol{y}}$ in support of the reflectivity.



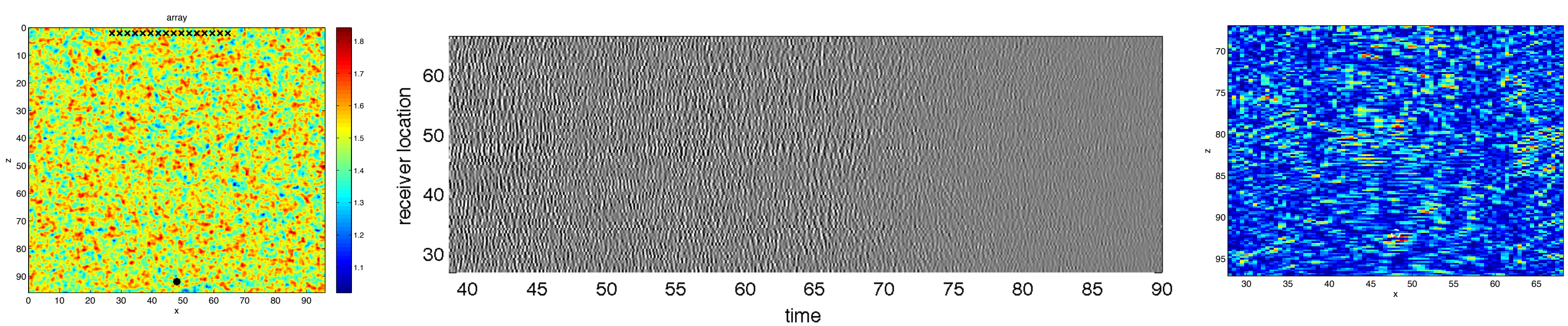
- Resolution in direction of propagation (range) is c_o/B. The pulse width $1/B$ determines precision of travel time estimation.

- Resolution in cross-range is $\sim \lambda_o L/a$.

Noise vs. clutter effects in migration imaging*



Noise



Noise is averaged out by summation (over large aperture).
Clutter is harder to deal with.

*Simulations by Chrysoula Tsogka.

Multiple scattering effects in media with strong reflectors*